*J. Fluid Mech.* (1975), *vol.* 71, *part* 4 *pp.* 729–754
*Printed in Great Britain*



# Linearized buoyant motion in a closed container

By M. C. JISCHKE
School of Aerospace, Mechanical and Nuclear Engineering, University of Oklahoma

AND R. T. DOTY
School of Engineering and Technology, McNeese State University

(Received 21 October 1974 and in revised from 24 February 1975)

An arbitrarily-shaped, closed container completely filled with fluid is considered. It is assumed that the fluid is originally in a stably-stratified state of rest, and that at an initial instant the temperature of the container walls is impulsively changed. The ensuing unsteady laminar motion is found by solving the linearized Boussinesq equations governing buoyancy-driven flows. A 'boundary-layer/inviscid-interior' decomposition leads to a modified asymptotic expansion scheme of analysis. The boundary-layer concept is valid only for large values of the Rayleigh number, and, in addition, we limit the Prandtl number to order unity. It is found that the inviscid interior region heats up by means of a convection process that is driven by suction induced by the boundary layer. The inviscid, adiabatic interior responds to a special horizontal 'average' value of the container temperature perturbation. The boundary layer smears out, or averages, any circumferential variation in this perturbation, so that the interior, in effect, responds to an isothermal boundary in each horizontal plane. The interior temperature and vertical velocity component are expressed simply in terms of this horizontal 'average' container temperature. The horizontal velocity potential is governed by a Poisson equation, whose solution is developed for several specific geometries to illustrate the nature of the flow.

## 1. Introduction

There are many engineering and scientific problems in which unsteady buoyancy-driven flows apparently play a central role. These problems can generally be classified as either 'external' problems, such as the flow around a heated rod or plate in an otherwise quiescent fluid, or 'internal' problems, such as the flow between parallel plates or in fluid-filled cavities.

As pointed out by Ostrach (1972), the external problems have received a great deal of attention, while relatively little has been done about internal problems thus far. Ostrach contends that the reason for this is not the greater importance of the external problem, but rather that internal natural-convection problems are considerably more complex. This is because those external natural-convection problems for which the Rayleigh number is large can be analysed by the usual Prandtl boundary-layer theory that is so helpful in other external fluid-flow problems. This is due to the assumption that the region exterior to the boundary layer can be assumed to be unaffected by the boundary layer in an external problem. For confined natural-convection problems, on the other hand, a boundary layer will exist near the walls (for sufficiently large Rayleigh number), but the region exterior to the boundary layer cannot be assumed to be independent of the boundary layer. In other words, the region exterior to the boundary layer will be completely enclosed by the boundary layer, and will form a core region that is greatly affected by boundary-layer behaviour. Hence, the boundary layer and core are closely coupled to each other; and this coupling constitutes the main source of difficulty in obtaining analytic solutions to internal problems.

There have been many efforts to analyse various models of the contained buoyancy-driven flow problem. Many of these efforts have used numerical finite-difference methods, and have been restricted to rectangular or cylindrical geometries. Few have focused on the important problem of transient flows. We shall restrict our attention to the practically important flow regime that corresponds to large values of the Rayleigh number (the ratio of buoyant forces to viscous forces); and we shall exclude the class of problems that deals with the stability of buoyancy-driven flows typified by the Rayleigh instability for heating from below. The Prandtl number is assumed to be of order unity; and we shall consider laminar flow exclusively.

Sakurai & Matsuda (1972), motivated by the work of Greenspan & Howard (1963), have considered a Boussinesq fluid at rest in a circular cylinder with its axis of symmetry parallel to the gravitational force. They analyse the unsteady flow that results from abruptly changing the side-wall temperature from its original profile, which is invariant around the cylinder (in a horizontal plane) and linear along the cylinder (in a vertical plane), to some new profile that is also linear along the cylinder. This new profile has a slightly greater rate of change than the original profile, and is oriented such that the container is heated above its horizontal mid-plane and cooled below its mid-plane. They apply a linearized theory for the case where the Prandtl number is of order unity, to find the temperature and velocity within the core region. They demonstrate the existence of a meridional circulation, which is pumped by a side-wall boundary layer. This meridional circulation redistributes the fluid, to bring about a new state of stratification. The time scale for this temperature adjustment process is shown to be given by the product of the one-fourth power of the Rayleigh number and the inverse of the Brunt–Väisälä frequency. This work by Sakurai & Matsuda is important, in that for the first time an analytical solution has been developed for the transient natural-convection flow of a fluid in a cavity that makes clear the basic physical processes that occur. The limitation to their work is that it is applied only to the circular cylinder geometry and a particular linear boundary condition. Thus, the general problem of linearized buoyant motion in a closed container remains to be solved.

Here we shall consider an arbitrarily-shaped closed container enclosing a Boussinesq fluid, which is initially in a stably-stratified state of rest. At an initial instant, the temperature of the container will be impulsively changed. We shall describe the ensuing unsteady laminar flow, as well as the final asymptotic steady state that is governed by the linearized equations. We refer to this problem as the general problem of heat-up from rest, since no restriction has been made as to the container temperature profile and, as we shall see, the container geometry is rather arbitrary. The solution to this general heat-up problem encompasses many linearized contained buoyant motions as special cases and, as such, should be of importance.

Those who study stratified fluids are aware that a very close analogy exists between stratified fluid phenomena and rotating fluid phenomena. For example, Veronis (1970) gives an extensive review of the analogy of between rotating and stratified fluids, and Greenspan (1969) often mentions the analogy in his discussion of rotating fluid theory. Barcilon & Pedlosky (1967) and Siegmann (1971), among others, have considered the effects of stratification in the linear theory of rotating fluids. The Coriolis force, however, plays a dominant role in these analyses, which are restricted to specific geometries. In contrast, it is our purpose here to focus attention on the behaviour of stratified fluids in arbitrarily-shaped containers and completely to ignore rotating fluid effects.

The rotating fluid problem that is similar to our heat-up problem for stratified fluids is the spin-down problem treated by Greenspan (1965). He considers an arbitrarily-shaped closed container filled with an incompressible fluid, which, at an initial instant, is in a physically acceptable initial state of fluid motion slightly different from a state of rigid rotation at the angular velocity of the container. Greenspan then analyses the ensuing transient motion, and describes the approach to the ultimate state of rigid-body rotation. We shall adapt the solution procedure used by Greenspan to solve our heat-up problem for a stratified fluid.

Briefly, the solution procedure is as follows. Expansion in half powers of the Ekman number (the fourth root of the reciprocal of the Rayleigh number is the appropriate parameter for stratified fluids) is introduced into the governing equations, and a problem sequence is resolved. The first of these problems is the zeroth-order solution for the inviscid interior region. In the second problem, the interior motion is corrected for viscous effects, to make the velocity zero at the boundary. However, the boundary layers cause further interior motion by inducing a small normal mass flux; and this sets up a third problem. Once the secondary interior motion is determined from this third problem, it too must be corrected at the boundary. The analysis ends with this secondary circulation, although in principle the procedure could be carried on to higher order. The goal of our work, then, is to adapt this solution procedure to the general heat-up problem for a stratified fluid, and to achieve an approximate solution for the motion that is uniformly valid in time and space.

## 2. Theoretical development

### 2.1. *Governing equations*

Consider an arbitrarily-shaped closed container $\Sigma$ which is completely filled with a 'Boussinesq' fluid, initially in a stable state of rest. At an initial instant $t = 0$, the temperature of $\Sigma$ is impulsively changed. A description of the unsteady flow that ensues is desired.

Assume that the fluid motion to be studied is a small perturbation on this basic state of stable static equilibrium. The following linear equations then apply (Doty 1973):

$$\nabla . \mathbf{V} = 0, \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla p + Ra^{-\frac{1}{2}}\nabla^2 \mathbf{V} + T\hat{\mathbf{e}}_z, \tag{2}$$

$$Pr\frac{\partial T}{\partial t} + W = Ra^{-\frac{1}{2}}\nabla^2 T. \tag{3}$$

$Pr$ and $Ra$ are the Prandtl number and Rayleigh number, respectively. $W$ is the vertical component of the total fluid velocity $\mathbf{V}$. Gravity $g$ is assumed to be constant and act in the negative $z$ direction.

Here the normalized pressure $p$ and temperature $T$ of the fluid are related to the physical pressure and temperature (dimensional quantities are denoted by an asterisk) according to

$$p = \frac{p^* - p_r^*}{\epsilon \bar{\rho}^* g^* L^* \bar{\beta}^* \Delta T^*}, \quad T = \frac{T^* - T_r^*}{\epsilon \Delta T^*}. \tag{4), (5}$$

The subscript $r$ refers to the initial, stable state of rest. The normalized fluid velocity $\mathbf{V}$, and time $t$ are given by

$$\mathbf{V} = \frac{\mathbf{V}^* Pr^{\frac{1}{2}}}{\epsilon L^* N}, \quad t = t^* N\, Pr^{\frac{1}{2}}. \tag{6), (7}$$

In these normalizations, $L^*$ is a typical container dimension in the vertical direction, and $\Delta T^*$ is the temperature stratification of the basic state across $L^*$. The order of the perturbation in the boundary temperature is $\epsilon \Delta T^*$, where $\epsilon \ll 1$. Reference values of the basic state (say, at $z = 0$) are over-scored, and $\beta$ represents the coefficient of thermal expansion of the fluid. The nonlinear terms that would appear in (2) and (3) can be shown to be of the order of $\epsilon$ or less everywhere in the container, and thus are neglected.

The non-dimensional parameters that appear in these equations are the Prandtl number, Rayleigh number and Brunt–Väisälä frequency:

$$Pr \equiv \bar{\mu}^* \bar{C}_p^* / \bar{k}^*, \quad Ra \equiv \bar{\rho}^{*2} g^* L^{*3} \bar{\beta}^* \Delta T^* \bar{C}_p^* / \bar{k}^* \bar{\mu}^*, \quad N \equiv (\bar{\beta}^* \Delta T^* g^* / L^*)^{\frac{1}{2}}.$$

$\mu^*$, $C_p^*$ and $k^*$ are the coefficients of viscosity, specific heat at constant pressure and thermal conductivity, respectively. These three parameters are assumed to be constant.

To complete the formulation of the problem, we add the following initial and boundary conditions: static equilibrium initially,

$$\mathbf{V}(\mathbf{r}, 0) = 0 \quad (t = 0), \tag{8}$$

given boundary temperature,

$$T(\mathbf{r}_\Sigma, t) = H(t)\, T_\Sigma(\mathbf{r}_\Sigma) \quad \text{on} \quad \Sigma, \tag{9}$$

no-slip on container,

$$\mathbf{V}(\mathbf{r}_\Sigma, t) = 0 \quad \text{on} \quad \Sigma, \tag{10}$$

bounded solution,

$$\mathbf{V}, T \text{ finite} \quad (t \to \infty). \tag{11}$$

Here $H(t)$ is the Heaviside step function, $T_\Sigma(\mathbf{r}_\Sigma)$ is the temperature profile on the boundary and $\mathbf{r}_\Sigma$ is the position vector to some point on the container.

### 2.2. *Solution procedure*

We shall now make several assumptions based primarily on work of Doty & Jischke (1974), which gives an exact solution to the problem of linearized natural convection flow due to an impulsively heated infinite vertical plate. We shall assume that the Rayleigh number is large, and that viscous action and heat conduction are confined to thin boundary layers at the container walls throughout the principal phase of the motion. These boundary layers entrain fluid, and produce a secondary motion in the inviscid, adiabatic interior that is of major importance in redistributing the internal energy in the interior to the new state of stratification. Buoyancy, in this fashion, adjusts the temperature to its steady-state value in the 'heat-up' time scale $Ra^{\frac{1}{4}}$, and $Ra^{-\frac{1}{4}}$ emerges as the significant expansion parameter.

An approximate solution is sought which consists of an inviscid, adiabatic motion throughout the interior of the container that is matched to a motion in the viscous, heat-conducting boundary layer, in order to satisfy the boundary conditions. Furthermore, the representation must be uniformly valid in time and space, to ensure that all the important phenomena are included and described. The solution procedure is to expand the flow variables in powers of $Ra^{-\frac{1}{4}}$, introduce these expansions into the governing equations, and resolve a problem sequence. The first of these problems is for the zeroth-order inviscid, adiabatic interior motion. In the second problem, the interior motion is corrected for viscous and conduction effects, to make the velocity zero at the boundary and to satisfy the boundary condition on the temperature. These boundary layers induce further interior motion by establishing a small mass flux normal to the boundary, which requires a third problem to correct the interior motion. The analysis ends with this first-order correction to the interior motion, although in principle, higher-order corrections could be carried out.

It is anticipated that the pressure force and the buoyant force will be in balance over many periods of the Brunt–Väisälä frequency within the inviscid, adiabatic interior and that the interior will change slowly, without oscillation, from its initial value to its final value on the heat-up time scale. Thus, the assumed form of the interior solution is given by

$$\mathbf{V} = \mathbf{V}(\mathbf{r}, \tau), \quad p = p(\mathbf{r}, \tau), \quad T = T(\mathbf{r}, \tau), \tag{12)–(14}$$

where

$$\tau = Ra^{-\frac{1}{4}}t. \tag{15}$$

This inviscid, adiabatic solution must be corrected for viscous action near the boundary, which in turn induces further motion in the interior, etc. The boundary layers produced by the temperature perturbation become fully developed in a relatively short time (a few periods of the Brunt–Väisälä frequency), then change very slowly during heat-up. Thus, as far as the inviscid motion is concerned, the boundary layers can be considered to be formed instantaneously, and to remain quasi-steady throughout the heat-up process. As a consequence, however, the initial condition on the velocity must be given up. The interior fluid will not be at rest initially, within the framework of this analysis, but will have some initial first-order motion that is dictated by the boundary-layer 'suction'. We thus lose the capability of exactly describing the boundary layers and associated secondary flow for the very earliest times, but only then. Furthermore, for the case of an impulsively-heated vertical plate (Doty & Jischke 1974), these quasi-steady boundary layers and secondary flows are substantially the same as those determined from an exact solution of the linearized equations.

Thus an approximate solution of the following form is sought:

$$\mathbf{V} = \mathbf{V}_0(\mathbf{r},\tau) + \tilde{\mathbf{V}}_0 + Ra^{-\frac{1}{4}}[\mathbf{V}_1(\mathbf{r},\tau) + \tilde{\mathbf{V}}_1] + \ldots, \tag{16}$$

$$p = p_0(\mathbf{r},\tau) + \tilde{p}_0 + Ra^{-\frac{1}{4}}[p_1(\mathbf{r},\tau) + \tilde{p}_1] + \ldots, \tag{17}$$

$$T = T_0(\mathbf{r},\tau) + \tilde{T}_0 + Ra^{-\frac{1}{4}}[T_1(\mathbf{r},\tau) + \tilde{T}_1] + \ldots. \tag{18}$$

Tilde denotes a boundary-layer function of a stretched boundary-layer co-ordinate $\xi$ (scaled by $Ra^{-\frac{1}{4}}$). These functions represent corrections to the inviscid solution in the boundary layer and approach zero exponentially fast as $\xi \to \infty$, corresponding to the outer edge of the boundary layer. The functions without tildes are then the solution of the inviscid, adiabatic equations of motion. The replacement of the dependent variables by such a decomposition leads to a modified system of equations within a singular-perturbation scheme of analysis.

Those familiar with the linearized theory of rotating fluids will recognize that our solution of the heat-up problem does not include inertial modes corresponding to oscillations on a time scale of order unity (corresponding to a dimensional time of the order of the inverse of the Brunt–Väisälä frequency). The expansions (16)–(18) are analogous to the geostrophic mode of linearized rotating fluid theory. We might, by analogy, refer to (16)–(18) as corresponding to the 'thermostrophic' mode of linearized buoyancy-driven fluid motion. The linear theory of rotating fluids represents the solution of the general spin-up problem as a superposition of a geostrophic mode and a presumably infinite number of inertial modes. Thus we should expect the solution of the linearized Boussinesq equations for the general heat-up problem to be similarly described as a superposition of thermostrophic and inertial modes. In the present case, however, the assumption of an initial state of rest means that only the thermostrophic mode is excited, and the inclusion of inertial modes is unnecessary. The more general case of an initial state of motion is being pursued.

The approximate form of the equations valid in the boundary layers can be obtained in the following manner. Let $\hat{\mathbf{n}}$ be defined as the outward-pointing unit normal to the container $\Sigma$. The fluid velocity in the boundary layer can then be resolved into components that are normal and tangential to $\Sigma$. This is expressed as

$$\tilde{\mathbf{V}} = (\tilde{\mathbf{V}}\,.\,\hat{\mathbf{n}})\,\hat{\mathbf{n}} - (\tilde{\mathbf{V}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}. \tag{19}$$

This expression can be substituted into the conservation equations; and, neglecting the tangential derivatives along $\Sigma$ of any flow variable compared with its normal derivative, we obtain the following set of equations, which are valid in the boundary layer:

$$\frac{\partial}{\partial \xi}(\tilde{\mathbf{V}}.\hat{\mathbf{n}}) + Ra^{-\frac{1}{4}}\hat{\mathbf{n}}.\nabla \times (\tilde{\mathbf{V}} \times \hat{\mathbf{n}}) = 0, \tag{20}$$

$$\frac{\partial \tilde{p}}{\partial \xi}\hat{\mathbf{n}} = Ra^{-\frac{1}{4}}\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla_2 \tilde{p} - \frac{\partial^2 \tilde{\mathbf{V}}}{\partial \xi^2} - \tilde{T}\hat{\mathbf{e}}_z\right], \tag{21}$$

$$Pr\frac{\partial \tilde{T}}{\partial t} + \tilde{W} = \frac{\partial^2 \tilde{T}}{\partial \xi^2}. \tag{22}$$

Here $\nabla_2$ is defined as the two-dimensional gradient operator with components in the plane of $\Sigma$. The formal development of these equations can proceed in several equivalent ways and the presentation here is taken from Greenspan (1969), who, in turn, based his work in part on that of Crabtree, Küchemann & Sowerby (1963).

### 2.3. *Problem sequence*

Substitution of the perturbation expansions into the governing equations and boundary conditions (1)–(3), (8)–(11) and (20)–(22) leads to a sequence of problems for the inviscid, adiabatic interior flow, the boundary-layer flow and their mutual interactions. The problem sequence is as follows.

*Problem* (i*a*), *the zeroth-order interior*:

$$\nabla.\mathbf{V}_0 = 0, \quad \nabla p_0 = T_0\hat{\mathbf{e}}_z, \quad W_0 = 0, \tag{23)–(25}$$

with boundary condition

$$\mathbf{V}_0.\hat{\mathbf{n}} = 0 \quad \text{on} \quad \Sigma. \tag{26}$$

*Problem* (i*b*), *the first-order boundary layer*:

$$\frac{\partial}{\partial \xi}(\tilde{\mathbf{V}}_1.\hat{\mathbf{n}}) = -\hat{\mathbf{n}}.\nabla \times (\tilde{\mathbf{V}}_0 \times \hat{\mathbf{n}}), \quad -\frac{\partial \tilde{p}_1}{\partial \xi}\hat{\mathbf{n}} = \frac{\partial^2 \tilde{\mathbf{V}}_0}{\partial \xi^2} + \tilde{T}_0\hat{\mathbf{e}}_z, \quad \tilde{W}_0 = \frac{\partial^2 \tilde{T}_0}{\partial \xi^2}, \tag{27)–(29}$$

with boundary conditions

$$\mathbf{V}_0 + \tilde{\mathbf{V}}_0 = 0 \quad \text{on} \quad \Sigma, \quad T_0 + \tilde{T}_0 = T_\Sigma \quad \text{on} \quad \Sigma. \tag{30), (31}$$

*Problem* (i*c*), *the first-order interior*:

$$\nabla.\mathbf{V}_1 = 0, \quad \frac{\partial \mathbf{V}_0}{\partial \tau} = -\nabla p_1 + T_1\hat{\mathbf{e}}_z, \quad Pr\frac{\partial T_0}{\partial \tau} + W_1 = 0, \tag{32)–(34}$$

with boundary conditions

$$\mathbf{V}_1 + \tilde{\mathbf{V}}_1 = 0 \quad \text{on} \quad \Sigma, \qquad T_0 = 0 \quad \text{at} \quad \tau = 0. \tag{35), (36}$$

The zeroth-order boundary-layer equations yield the trivial results

$$\tilde{\mathbf{V}}_0.\hat{\mathbf{n}} = \tilde{p}_0 = 0, \tag{37}$$

which have been incorporated into the first-order problem. These results follow from the fact that the zeroth-order equations show that $\tilde{\mathbf{V}}_0.\hat{\mathbf{n}}$ and $\tilde{p}_0$ are constants and these constants must be zero if the boundary-layer corrections are to decay to zero exponentially in $\xi$.

The vorticity equation for the inviscid interior can be used to show that the vertical component of vorticity is always zero to all orders of approximation. To see this, take the curl of the momentum equation (12), to obtain an equation for the vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{V}$:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = Ra^{-\frac{1}{2}}\nabla^2\boldsymbol{\omega} + \nabla T \times \hat{\mathbf{e}}_z. \tag{38}$$

If we neglect the viscous term, the vertical component of (38) is

$$\frac{\partial}{\partial t}(\hat{\mathbf{e}}_z \cdot \boldsymbol{\omega}) = 0. \tag{39}$$

Thus, we see that the vertical component of vorticity in the inviscid interior is constant for all time and, since the fluid starts from a state of rest there, that constant must be zero. Hence, the fluid will never contain any vertical vorticity in the inviscid interior (unless, of course, vertical vorticity is created in the boundary layer and diffused into the interior on the time scale of $Ra^{-\frac{1}{2}}$). This is true before, during and after the initial instant in which the boundary layers form. This result is crucial to the analysis that follows.

## 3. Analysis

The analysis of the heat-up from rest problem requires solving problems (i$a$)–(i$c$) for the interior flow, the boundary-layer correction, and their mutual interaction. The solution of these problems will be taken up now.

### 3.1. *Zeroth-order interior*

Problem (i$a$), which describes the zeroth-order interior flow, is given by (23)–(26). The curl of the momentum equation (24) yields $\nabla T_0 \times \hat{\mathbf{e}}_z = 0$, which shows that $T_0$ depends only on the vertical spatial co-ordinate $z$. Of course, time enters into the description of the interior flow, but only as a parameter. Further analysis of the zeroth-order interior must be deferred until problem (i$c$) is considered.

### 3.2. *First-order boundary layer*

The first-order boundary layer is described by problem (i$b$), which is given by (27)–(31). The scalar components of the momentum equation (28), normal to the container and in the vertical direction, are

$$-\frac{\partial \tilde{p}_1}{\partial \xi} = \tilde{T}_0(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z), \quad -\frac{\partial \tilde{p}_1}{\partial \xi}(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z) = \frac{\partial^2 \tilde{W}_0}{\partial \xi^2} + \tilde{T}_0, \tag{40), (41}$$

where we have used the condition $\tilde{\mathbf{V}}_0 \cdot \hat{\mathbf{n}} = 0$ from (26) and (36). By combining these two scalar momentum equations with the energy equation, we derive a single fourth-order equation for the boundary-layer temperature, which may be written as

$$\partial^4 \tilde{T}_0 / \partial \xi^4 + [1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z)^2]\, \tilde{T}_0 = 0. \tag{42}$$

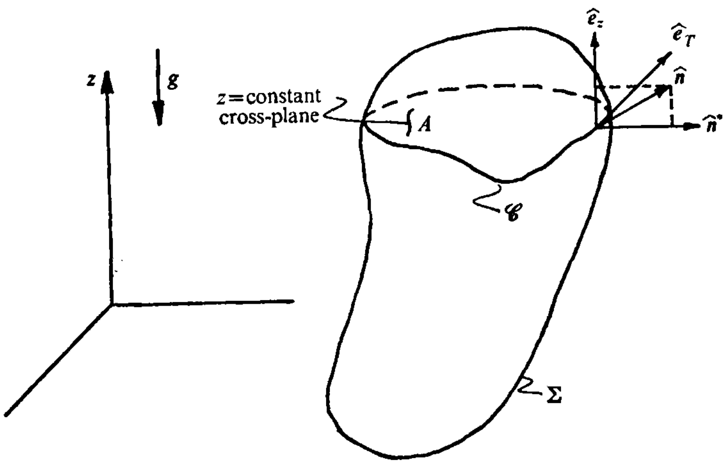


FIGURE 1. Cross-plane geometry and unit vectors.

It is convenient to rescale the boundary-layer variable $\xi$, as follows (assuming $1-\hat{\mathbf{n}}\,.\,\hat{\mathbf{e}}_z$ is not of the order of $Ra^{-\frac{1}{4}}$, or smaller):

$$\xi^* \equiv \frac{[1-(\hat{\mathbf{n}}\,.\,\hat{\mathbf{e}}_z)^2]^{\frac{1}{4}}}{4}\,\xi. \tag{43}$$

Then the equation for the boundary-layer temperature becomes simply

$$\frac{\partial^4 \tilde{T}_0}{\partial \xi^{*4}} + 4\tilde{T}_0 = 0. \tag{44}$$

The no-slip condition and the condition that $\tilde{T}_0$ must decay exponentially with $\xi^*$, allow one to obtain the solution for $\tilde{T}_0$

$$\tilde{T}_0 = \tilde{T}_0(\mathbf{r}_\Sigma;\tau)\exp(-\xi^*)\cos\xi^*. \tag{45}$$

$\mathbf{r}_\Sigma$ is the position vector to some point on the container; $\tilde{T}_0(\mathbf{r}_\Sigma;\,\tau)$ is the boundary-layer temperature on the container.

As we have seen, the component of the zeroth-order boundary-layer velocity normal to the container wall must vanish, so that the entire zeroth-order boundary-layer velocity is tangential to the container. Hence, the vector momentum equation that describes the boundary-layer flow in a direction tangential to the container is, from (28),

$$\partial^2\tilde{\mathbf{V}}_0/\partial\xi^2 = \tilde{T}_0(\hat{\mathbf{e}}_z\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}. \tag{46}$$

The expression just found for the temperature can be substituted into this equation. Then, by integrating twice and using the fact that the boundary-layer velocity must decay exponentially across the boundary layer, the zeroth-order boundary-layer velocity is found to be

$$\tilde{\mathbf{V}}_0 = \tilde{T}_0(\mathbf{r}_\Sigma;\tau)\left[\frac{(\hat{\mathbf{n}}\times\hat{\mathbf{e}})\times\hat{\mathbf{n}}}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|}\right]\exp(-\xi^*)\sin\xi^*. \tag{47}$$

Let $\mathscr{C}$ be the curve formed by intersecting $\Sigma$ with a horizontal 'cross-plane', such that $\mathscr{C}$ bounds the cross-plane area $A$ as shown in figure 1. Define $\hat{\mathbf{n}}^*$ to be the normalized component of the container's outward pointing unit normal $\hat{\mathbf{n}}$, such that $\hat{\mathbf{n}}^*$ lies in the cross-plane. Then $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}^*$ are related by

$$\hat{\mathbf{n}} = |\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z| \, \hat{\mathbf{n}}^* + (\hat{\mathbf{n}} \, . \, \hat{\mathbf{e}}_z) \, \hat{\mathbf{e}}_z. \tag{48}$$

Define $\hat{\mathbf{e}}_T$ to be a unit vector tangential to $\mathscr{C}$ in the cross-plane. Then $\hat{\mathbf{e}}_T$ is related to $\hat{\mathbf{n}}$ by

$$\hat{\mathbf{e}}_T = \hat{\mathbf{e}}_z \times \hat{\mathbf{n}} / |\hat{\mathbf{e}}_z \times \hat{\mathbf{n}}|, \tag{49}$$

and $(\hat{n}^*, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_z)$ form an orthonormal triad on the container. Thus, in terms of $\hat{\mathbf{e}}_T$, we may rewrite (47) as

$$\tilde{\mathbf{V}}_0 = \tilde{T}_0(\mathbf{r}_\Sigma; \tau) \, (\hat{\mathbf{n}} \times \hat{\mathbf{e}}_T) \exp(-\xi^*) \sin \xi^*. \tag{50}$$

Evaluating $\tilde{\mathbf{V}}_0 \times \hat{\mathbf{n}}$, and integrating the continuity equation (27) across the boundary layer, we obtain an expression for the first-order normal component of the boundary-layer velocity on the container. This boundary-layer 'suction' is given by

$$(\tilde{\mathbf{V}}_1 \, . \, \hat{\mathbf{n}})_\Sigma = \frac{1}{\sqrt{2}} \hat{\mathbf{n}} \, . \, \nabla \times \left[ \frac{\tilde{T}_0(\mathbf{r}_\Sigma; \tau) \, \hat{\mathbf{e}}_T}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}|^{\frac{1}{2}}} \right]. \tag{51}$$

Thus, the boundary-layer suction is solved to within a function that represents the boundary-layer correction to the temperature distribution on the container.

From (50), we see that the lowest-order boundary-layer correction to the velocity has no component in the direction of $\hat{\mathbf{e}}_T$. Thus the circulation of $\tilde{\mathbf{V}}_0$ around any curve parallel to $\mathscr{C}$ is zero and, by Stokes's theorem, there is no vertical vorticity created in the boundary layer to lowest order. We can therefore conclude that, not only does the inviscid, adiabatic interior possess no vertical vorticity initially and during the heat-up time, but, even on the longer diffusion time scale, there will be no vertical vorticity in the interior up to and including order $Ra^{-\frac{1}{4}}$.

### 3.3. *First-order interior*

Problem (i*c*) describes the first-order interior flow; and it is given by (32)–(36). To proceed, we first decompose the interior velocity vector into a component in the cross-plane and a component normal to it. In other words, let

$$\mathbf{V} = \mathbf{v} + W \hat{\mathbf{e}}_z, \tag{52}$$

where, by definition, $\mathbf{v}$ has no component in the $z$ direction. If $\nabla_2$ is the two-dimensional gradient operator in the cross-plane, then the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{V}$ has a vertical component $\hat{\mathbf{e}}_z \, . \, \boldsymbol{\omega} = |\nabla_2 \times \mathbf{v}|$. Thus, the magnitude of the cross-plane vorticity is equal to the $z$ component of the total vorticity to every order. But, as seen from (39) and (50), this component of the total vorticity is zero for all times up to and including the diffusion time, and for all orders up to and including $Ra^{-\frac{1}{4}}$. Thus, we conclude that the cross-plane velocity is irrotational in the cross-plane, up to at least order $Ra^{-\frac{1}{4}}$.

The continuity equation states that the total velocity vector has no divergence to all orders. As a direct consequence of this, we may use (52) to write

$$\nabla_2 \, . \, \mathbf{v} = -\partial W / \partial z. \tag{53}$$

This can be interpreted as the continuity equation for a constant-density two-dimensional flow in the cross-plane, with a source term due to the vertical motion into the cross-plane.

The zeroth-order interior has no flow in the vertical direction, as shown by (25). Thus, the zeroth-order cross-plane velocity has not only zero two-dimensional curl, but also zero two-dimensional divergence. It follows that the zeroth-order flow is a potential flow in the cross-plane. Furthermore, the normal component of the zeroth-order velocity vanishes on the container, as seen from (26). Hence, the zeroth-order flow in the cross plane is described by the Neumann problem for Laplace's equation with a vanishing boundary condition. Thus, the cross-flow vanishes, and we conclude that the inviscid interior is motionless to zeroth order.

Since the first-order velocity has a non-zero vertical component, the equation (53) for the first-order cross-flow becomes

$$\nabla_2 . \mathbf{v}_1 = -\partial W_1/\partial z. \tag{54}$$

We may integrate (54) over the cross-plane area $A$, and use the energy equation (34) to show that, like $T_0$, $W_1$ depends on only the vertical spatial co-ordinate (as well as time $\tau$, of course). The result of this integration is

$$\iint_A \nabla_2 . \mathbf{v}_1 \, dA = -A\, \partial W_1/\partial z. \tag{55}$$

Green's theorem in the plane may be used to convert the area integral into a line integral. Then, substituting for $\hat{\mathbf{n}}^*$ in terms of $\hat{\mathbf{n}}$ and writing $\mathbf{v}_1 = \mathbf{V}_1 - W_1 \hat{\mathbf{e}}_z$, we can rewrite (55) as

$$A\frac{\partial W_1}{\partial z} + \left[\oint_{\mathscr{C}} \frac{-\hat{\mathbf{n}} . \hat{\mathbf{e}}_z}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|} ds\right] W_1 = -\oint_{\mathscr{C}} \frac{\mathbf{V}_1 . \hat{\mathbf{n}}}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|} ds. \tag{56}$$

Leibnitz's rule for differentiating an integral can be used to show that the line integral on the left side of this equation is nothing more than the derivative of the cross-plane area $A$. The argument of the line integral on the right is directly related to the boundary-layer suction through the boundary condition given by (30). We eliminate $W_1$ in favour of $T_0$ by using (34), and substitute the expression for the boundary-layer suction given by (51) into (56), to obtain, after converting the line integral in (56) to a surface integral over $\Sigma_T$ (that portion of the container surface $\Sigma$ that lies above $\mathscr{C}$),

$$\frac{\partial}{\partial z}\left\{A Pr \frac{\partial T_0}{\partial \tau} - \frac{1}{\sqrt{2}} \iint_{\Sigma_T} \hat{\mathbf{n}} . \nabla \times \left[\frac{\tilde{T}_0(\mathbf{r}_\Sigma; \tau)\, \hat{\mathbf{e}}_T}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|^{\frac{1}{2}}}\right] d\Sigma\right\} = 0. \tag{57}$$

Using Stokes's theorem to convert the surface integral to a line integral, and making use of the fact that the temperature is specified on the container, so that

$$\tilde{T}_0(\mathbf{r}_\Sigma; \tau) = T_\Sigma - T_0 \quad \text{on} \quad \Sigma, \tag{58}$$

we can rewrite (57), after integration, as

$$A Pr \frac{\partial T_0}{\partial \tau} + \left[\frac{1}{\sqrt{2}} \oint_{\mathscr{C}} \frac{ds}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|^{\frac{1}{2}}}\right] T_0 = \frac{1}{\sqrt{2}} \oint_{\mathscr{C}} T_\Sigma \frac{ds}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|^{\frac{1}{2}}} + f(\tau), \tag{59}$$

where $f(\tau)$ is the function of integration. Since $T_\Sigma$ depends on spatial location and $f(\tau)$ does not, it is obvious that $f(\tau)$ is independent of $T_\Sigma$. That is, $f(\tau)$ is the same no matter what temperature boundary condition is imposed. Furthermore, if $T_\Sigma$ remains zero for all time, then $T_0$ must remain zero for all time, which leads to the conclusion that $f(\tau)$ is identically zero. Hence, the solution to (63) can be obtained by use of the integrating factor as

$$T_0 \exp\left[\frac{1}{\sqrt{2}\,Pr}\frac{1}{A}\oint_{\mathscr{C}}\frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}}\tau\right] = \frac{1}{\sqrt{2}\,Pr}\frac{1}{A}\oint_{\mathscr{C}} T_\Sigma \frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}} \times \exp\left[\frac{1}{\sqrt{2}\,Pr}\frac{1}{A}\oint_{\mathscr{C}}\frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}}\tau\right]\left[\frac{1}{\sqrt{2}\,Pr}\frac{1}{A}\oint_{\mathscr{C}}\frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}}\right]^{-1} + g(z), \quad (60)$$

where $g(z)$ is the function of integration.

Let us define the 'average' value of any variable $Q$ around $\mathscr{C}$ to be

$$\langle Q\rangle \equiv \oint_{\mathscr{C}} Q\frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}}\left[\oint_{\mathscr{C}}\frac{ds}{|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}}\right]^{-1}. \quad (61)$$

Thus, the $e$-folding time $\tau_H$ can be defined as

$$\tau_H \equiv \sqrt{2}\,Pr\,A/\mathscr{C}\langle|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}\rangle, \quad (62)$$

so that (60) may be rewritten as

$$T_0 = \langle T_\Sigma\rangle\,[1-\exp(-\tau/\tau_H)], \quad (63)$$

where we have determined $g(z)$ such that $T_0$ is initially zero, as required by (36). The solution for the first-order vertical velocity component in the interior is given in terms of the time derivative of this temperature by the energy equation (34). This velocity component can thus be written as

$$W_1 = -Pr/\tau_H\langle T_\Sigma\rangle \exp(-\tau/\tau_H). \quad (64)$$

Thus the interior temperature and vertical velocity depend on only the averaged container temperature $\langle T_\Sigma\rangle$. This result is similar to that obtained by Barcilon & Pedlosky (1967) for the interior motion of a stratified fluid in a differentially rotated right circular vertical cylinder. In that case, the cylinder's temperature was arbitrarily prescribed, and the top circular plate and side wall of the cylinder rotated at a constant angular velocity, while the lower circular plate had a slightly different angular velocity. Using the linear theory of rotating, stratified flows, they found that the steady interior motion was horizontal and depended on only $\langle T_\Sigma\rangle$. At a steady state, (64) shows the interior motion for flows driven exclusively by buoyancy from a state of rest must also be horizontal, although, as we shall see for our case of non-rotating, stratified flows, the interior motion does not depend on $\langle T_\Sigma\rangle$ only.

The non-dimensional $e$-folding time, or 'heat-up' time $\tau_H$, can be rewritten in dimensional form as

$$t_H = \sqrt{2}\frac{A}{\mathscr{C}}\frac{Pr^{\frac{1}{2}}\,Ra^{\frac{1}{4}}}{N}\langle|\hat{\mathbf{n}}\times\hat{\mathbf{e}}_z|^{\frac{1}{2}}\rangle, \quad (65)$$

where $t_H$ is the dimensional heat-up time, $A$ the horizontal cross-sectional area of the container non-dimensionalized by $L^2$, $\mathscr{C}$ the circumference of $A$ non-dimensionalized by $L$, $L$ the height of the container, $N$ the Brunt–Väisälä frequency, $Pr$ the Prandtl number, $Ra$ the Rayleigh number.

To complete the solution to the general heat-up problem, we must find the first-order interior flow in the cross-plane. As we have already seen, the first-order cross-plane velocity is irrotational in the cross-plane (i.e. $\nabla_2 \times \mathbf{v}_1 = 0$). Thus, we may derive the cross-plane velocity from a scalar potential $\phi_1$ by writing $\mathbf{v}_1 = \nabla_2 \phi_1$. Therefore, from (54), we obtain the Poisson equation for the cross-plane velocity potential $\phi_1$,

$$\nabla_2^2 \phi_1 = -\partial W_1/\partial z. \tag{66}$$

The boundary condition given by (35), when dotted with $\hat{\mathbf{n}}$, can be rewritten by means of (48) and (52) as

$$(\mathbf{v}_1 + W_1 \hat{\mathbf{e}}_z) . (|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z| \, \hat{\mathbf{n}}^* + (\hat{\mathbf{n}} . \hat{\mathbf{e}}_z) \hat{\mathbf{e}}_z) = -\tilde{\mathbf{V}}_1 . \hat{\mathbf{n}} \quad \text{on} \quad \Sigma. \tag{67}$$

Substituting for the boundary-layer suction from (51), and rewriting $\mathbf{v}_1$ in terms of the velocity potential $\phi_1$, gives the boundary condition for (66):

$$\frac{\partial \phi_1}{\partial n^*} = -\frac{(\hat{\mathbf{n}} . \hat{\mathbf{e}}_z)}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|} W_1 - \frac{1}{\sqrt{2}} \frac{\hat{\mathbf{n}}}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|} . \nabla \times \left\{ \frac{(T_\Sigma - T_0) \hat{\mathbf{e}}_T}{|\hat{\mathbf{n}} \times \hat{\mathbf{e}}_z|^{\frac{1}{2}}} \right\} \quad \text{on} \quad \Sigma, \tag{68}$$

where we have used (31) to express $\tilde{T}_0(\mathbf{r}_\Sigma; \tau)$ in terms of $T_\Sigma$ and $T_0$. Thus, we find that the flow in the cross-plane is equivalent to the flow of a two-dimensional, constant density, irrotational fluid, with a uniform source distribution provided by the vertical velocity. The boundary condition reflects the fact that fluid penetrates the cross-plane circumference, because of changing cross-plane area and boundary-layer suction.

This completes the lowest-order solution to the general heat-up problem. The temperature and vertical velocity are completely specified in terms of the 'average' value around $\mathscr{C}$ of the container temperature. The cross-flow solution follows from the above Poisson equation and associated boundary condition, which, of course, will depend upon the particular container geometry and temperature boundary condition being considered.

### 3.4. *Remarks*

It is worth noting that, unlike the usual external forced-convection boundary-layer theory, the theory of buoyancy-driven contained fluids has an inherent coupling between the inviscid region and the associated internal boundary layers. In other words, we cannot calculate the lowest-order interior motion, make a boundary-layer correction, then proceed to calculate the first-order interior, etc. Instead, each 'interior boundary layer' pair must be calculated simultaneously. For example, the equation for the zeroth-order interior temperature (57) contains imbedded in it the zeroth-order boundary-layer temperature, which in turn is known only in terms of the interior temperature. This reflects the fact that the lowest-order inviscid flow is driven by the lowest-order boundary layer, a situation far different from the more familiar external forced-convection problem.

We can also write the energy equation for the next higher-order interior flow. It is

$$Pr\,\partial T_1/\partial\tau + W_2 = -\nabla^2 T_0. \tag{69}$$

We see that, for the final steady state, this becomes

$$W_2 = -d^2\langle T_\Sigma\rangle/dz^2. \tag{70}$$

From (70) we can note the difference between a flow driven by a boundary temperature linear in $z$, and one driven by a nonlinear boundary temperature. In the first case, the vertical motion ceases when thermodynamic equilibrium is achieved, and the boundary layers die out in the heat-up time. In the second case, the final steady temperature profile is incompatible with a state of static equilibrium, and, although the lowest-order vertical motion ceases in the heat-up time, the next higher-order verticity component $W_2$ apparently does not.

The boundary-layer approximation employed in this development may require modification in cases where sharp corners exist. Further, the analysis of the first-order interior motion tacitly assumes the continuity of the derivative of the cross-sectional area $A(z)$. Discontinuities in $dA/dz$ (such as occur in the annular region between two spheres) may give rise to internal shear layers. Thus, it is to be understood that, in referring to arbitrarily-shaped closed containers here, we are excluding containers with sharp corners, or discontinuities in cross-sectional area that give rise to internal shear layers. These problems require further study.

The role of boundary layers on horizontal surfaces (for which $\hat{\mathbf{e}}_z . \hat{\mathbf{n}} = 1$) has also been ignored in this discussion of the general problem of heat-up from rest. As the analogy with the linear theory of rotating fluids suggests, horizontal boundary layers with thickness of the order of $Ra^{-\frac{1}{6}}$ would be expected to arise on horizontal surfaces connecting non-horizontal surfaces to ensure global mass conservation by transporting mass from one $Ra^{-\frac{1}{4}}$ buoyancy-driven boundary layer to another. Also, horizontal boundary layers with thickness of order $Ra^{-\frac{1}{3}}$ would be expected to occur, to smooth out any abrupt changes or discontinuities in temperature. Analysis shows this is indeed the case. A detailed description of the results for these horizontal boundary layers at this time, however, would take us too far astray.

The present theory also assumes the nonlinear convection terms, of order $\epsilon$, are negligible. More precisely, in the interior terms of order $\epsilon$ have been deleted. This then implies the restriction $\epsilon < Ra^{-\frac{1}{4}}$. If, however, $T_\Sigma \neq \langle T_\Sigma\rangle$, then a more stringent condition applies. In this case, buoyancy-driven boundary layers occupying an area of order $Ra^{-\frac{1}{4}}$ persist in time with a circumferential vorticity (made dimensionless with respect to $\epsilon N$) of order $Ra^{\frac{1}{4}}$. Owing to nonlinear vortex twisting, this gives rise to a vertical vorticity of order $\epsilon\, Ra^{\frac{1}{2}}$. Provided these boundary layers persist over a diffusion time $Ra^{\frac{1}{2}}$, this vertical vorticity will diffuse into the interior, giving rise to a vertical vorticity there of order $\epsilon\, Ra^{\frac{1}{4}}$. This nonlinear contribution to the interior vorticity has been neglected in the present development. Such an approximation is valid, provided $\epsilon\, Ra^{\frac{1}{4}} < Ra^{-\frac{1}{4}}$. Thus, in the case $T_\Sigma \neq \langle T_\Sigma\rangle$, the more stringent condition $\epsilon < Ra^{-\frac{1}{2}}$ must prevail, if the results of this analysis are to be valid.

## 4. Some examples

The solution to the general problem of heat-up from a state of rest of a Boussinesq fluid in an arbitrarily-shaped closed container is complete, as far as the lowest-order temperature and vertical velocity are concerned. Given a particular container geometry and temperature boundary condition, one may, from (63) and (64), immediately write down the solutions for temperature and vertical velocity, in terms of the heat-up time $\tau_H$ and the 'average' value of the boundary temperature $\langle T_\Sigma \rangle$.

From (63), we see that the inviscid interior temperature varies only with vertical position within the container, and that this interior temperature approaches the average value of the container temperature asymptotically in time. Furthermore, we see from (64) that the vertical component of velocity in the interior approaches zero asymptotically in time. Thus, if there is any motion at all in the final steady state, that motion must be purely horizontal. This tendency toward horizontal flow is a characteristic of all stratified fluids, and is often referred to as 'plugging' or 'plugged' flow. This phenomenon may be predicted by simply inspecting the linearized energy equation (3). This equation shows that $W$ is identically zero in the inviscid interior when the fluid is at steady state.

The problem for the irrotational flow in the cross-plane, as defined by the elliptic Poisson equation (66) with the boundary condition given by (68), is a well-posed problem, whose solution for any container geometry and temperature perturbation is straightforward. Furthermore, there are special situations for which closed-form analytic solutions can be found. We shall now attempt to reveal the important physical notions associated with various container geometries and boundary temperatures, by treating several of these analytical examples.

### 4.1. *Right vertical cylinders*

We first consider the special class of containers consisting of right vertical cylinders of arbitrary cross-section. For this class of containers, we have the two conditions

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z = 0, \quad \hat{\mathbf{n}} \times \hat{\mathbf{e}}_z = 1, \tag{71}$$

for the side walls. In this case, (61) yields the result

$$\langle T_\Sigma \rangle = \mathscr{C}^{-1} \oint_{\mathscr{C}} T_\Sigma \, ds, \tag{72}$$

which we recognize as the conventional average value of $T_\Sigma$ around $\mathscr{C}$. Furthermore, the heat-up time is a constant given by (62):

$$\tau_H = \sqrt{2}\, Pr\, A/\mathscr{C}. \tag{73}$$

Thus, the temperature and vertical velocity solutions for all right vertical cylinders are given by (63) and (64):

$$T_0 = \langle T_\Sigma \rangle [1 - \exp(-\tau/\tau_H)], \tag{74}$$

$$W_1 = -\frac{1}{\sqrt{2}} \frac{\mathscr{C}}{A} \langle T_\Sigma \rangle \exp(-\tau/\tau_H), \tag{75}$$

where $\langle T_\Sigma\rangle$ is the conventional average value of the boundary temperature around the container perimeter $\mathscr{C}$, and $\tau_H$ is a constant that depends on the Prandtl number and the ratio of the container's cross-sectional area to its perimeter.

The cross-flow problem for all right vertical cylinders becomes

$$\nabla_2^2\phi_1 = \frac{1}{\sqrt{2}}\frac{\mathscr{C}}{A}\frac{\partial}{\partial z}\langle T_\Sigma\rangle \exp(-\tau/\tau_H), \tag{76}$$

with boundary condition

$$\frac{\partial\phi_1}{\partial n} = -\frac{1}{\sqrt{2}}\hat{\mathbf{n}}\,.\nabla\times[(T_\Sigma-\langle T_\Sigma\rangle+\langle T_\Sigma\rangle\exp(-\tau/\tau_H))\hat{\mathbf{e}}_T] \quad \text{on} \quad \mathscr{C}. \tag{77}$$

From this problem for flow in the cross-plane, we deduce the important result that the final steady-state motion in the cross-plane is always zero through first order, unless the temperature boundary condition varies around the cross-plane perimeter $\mathscr{C}$. In other words, as $\tau\to\infty$ we have

$$\nabla_2^2\phi_1\to 0, \tag{78}$$

with

$$\frac{\partial\phi_1}{\partial n}\to -\frac{1}{\sqrt{2}}\hat{\mathbf{n}}\,.\nabla\times[(T_\Sigma-\langle T_\Sigma\rangle)\hat{\mathbf{e}}_T] \quad \text{on} \quad \mathscr{C}. \tag{79}$$

This equation describes a non-zero motion only when $T_\Sigma$ differs from its average value around $\mathscr{C}$. Thus, a steady non-zero cross-flow results only if the prescribed container temperature perturbation varies around the perimeter of the container.

### 4.2. *Right circular vertical cylinders*

If we speciali e the cross-section of the right vertical cylinder of height $L$ to be a circle of radius $R$, then the non-dimensional cross-plane area $A$ equals $\pi(R/L)^2$, and the circumference $\mathscr{C}$ equals $2\pi(R/L)$. Hence, the heat-up time, as given by (73), is

$$\tau_H = \frac{Pr}{\sqrt{2}}\frac{R}{L}. \tag{80}$$

Now assume that the given boundary temperature $T_\Sigma$ depends on only the vertical spatial co-ordinate $z$ (i.e. $T_\Sigma = T_\Sigma(z)$). We refer to this as a $z$-dependent boundary condition, in contrast to one which depends upon both $z$ and some azimuthal co-ordinate. Since $T_\Sigma$ does not vary around $\mathscr{C}$, it is obvious that the average value of $T_\Sigma$ around $\mathscr{C}$ is equal to $T_\Sigma$ itself. Of course, this can be seen directly from (72); and it can be expressed as $\langle T_\Sigma\rangle = T_\Sigma$. This result holds for any $z$-dependent boundary condition, regardless of container geometry. Thus, the temperature and vertical velocity solutions may be written down directly from (74) and (75) as

$$T_0 = T_\Sigma[1-\exp(-\tau/\tau_H)], \tag{81}$$

$$W_1 = -\sqrt{2}\frac{L}{R}T_\Sigma \exp(-\tau/\tau_H), \tag{82}$$

where $\tau_H$ is given by (80). The Poisson problem for the velocity potential $\phi_1$ is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\phi_1}{dr}\right) = \sqrt{2}\frac{L}{R}\frac{dT_\Sigma}{dz}\exp(-\tau/\tau_H), \tag{83}$$

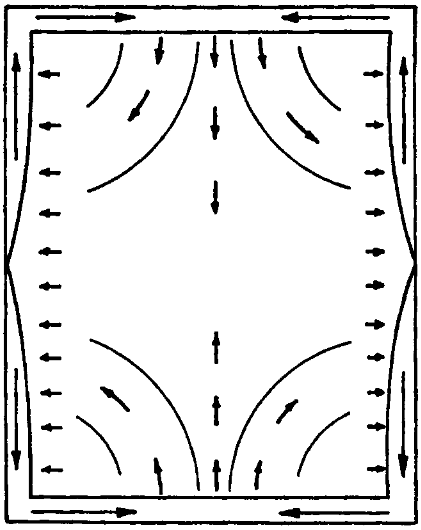


FIGURE 2. Four-celled flow produced by linear boundary temperature. $T_\Sigma = z - \frac{1}{2}$. Cross-section taken through centre-line.

with boundary condition

$$\frac{d\phi_1}{dr} = \frac{1}{\sqrt{2}} \frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H) \quad \text{on} \quad r = \frac{R}{L}, \tag{84}$$

where we have reasoned that $\phi_1 \neq \phi_1(\theta)$ by symmetry, and reduced (76) and (77) to this form. The solution to this problem is found, by a straightforward integration and application of the boundary condition, to be

$$\frac{d\phi_1}{dr} = \frac{1}{\sqrt{2}} \frac{L}{R} r \frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H). \tag{85}$$

Thus, we obtain the following expressions for the cross-plane velocity components in cylindrical polar co-ordinates:

$$u_1 = \frac{1}{\sqrt{2}} \frac{L}{R} r \frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H), \quad v_1 = 0. \qquad (86), (87)$$

Since there is no flow in the azimuthal direction, it follows that the streamlines are along lines of constant $\theta$. Hence, we see that the flow in the cross-plane is purely radial, and that the speed of this cross-plane flow increases with increasing distance from the centre-line of symmetry of the circular cylinder.

For the purpose of illustration, let us consider a boundary temperature that varies linearly in $z$, $T_\Sigma = z - \frac{1}{2}$. Here, as sketched qualitatively in figure 2, the fluid rises near the heated part of the boundary ($z > \frac{1}{2}$). Hence, fluid is always being entrained from the interior into the buoyancy-driven viscous boundary layers, and they grow as shown. This horizontal entrainment causes a weak vertical motion in the interior as a consequence of mass conservation. In the upper half of the cylinder, fluid particles in the interior descend, while they rise in the lower half of the cylinder. As the interior is inviscid and adiabatic, the tempera-

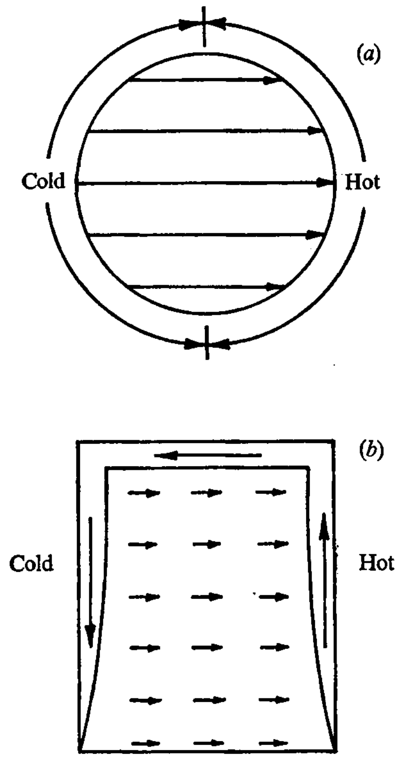


FIGURE 3. Qualitative sketch of flow produced by azimuthally-varying boundary temperature. (a) Top view of interior streamlines. (b) Cross-section taken through centre-line. $T_\Sigma = f(z)\cos 2\theta$.

ture of a fluid particle there remains constant. Particles in the upper (lower) half of the cylinder continue to fall (rise) until they reach a level where their temperature equals the heated (cooled) wall temperature. At this point the motion ceases, the boundary layers decay, and the fluid is heated. This particular geometry and boundary condition was studied by Sakurai & Matsuda (1972) following a different method; and our solution agrees exactly with theirs.

Let us now assume that the container temperature is known to be

$$T_\Sigma = f(z)\cos(\alpha\theta), \tag{88}$$

where $\alpha$ must be an integer so that $T_\Sigma$ is single-valued. Then its average value, as given by (72), is zero. From (74) and (75) we conclude that both $T_0$ and $W_1$ vanish. This result illustrates what is apparent from inspecting (74) and (75): namely, that the interior heat-up process is a response to the 'average' value of the boundary temperature, and when that average is zero the fluid does not heat up. However, this does not mean there is no cross-flow. We can see this by inspecting the appropriate Poisson equation for this example, which reduces to

$$r^2\frac{\partial^2\phi_1}{\partial r^2}+r\frac{\partial\phi_1}{\partial r}+\frac{\partial^2\phi_1}{\partial\theta^2} = 0, \tag{89}$$

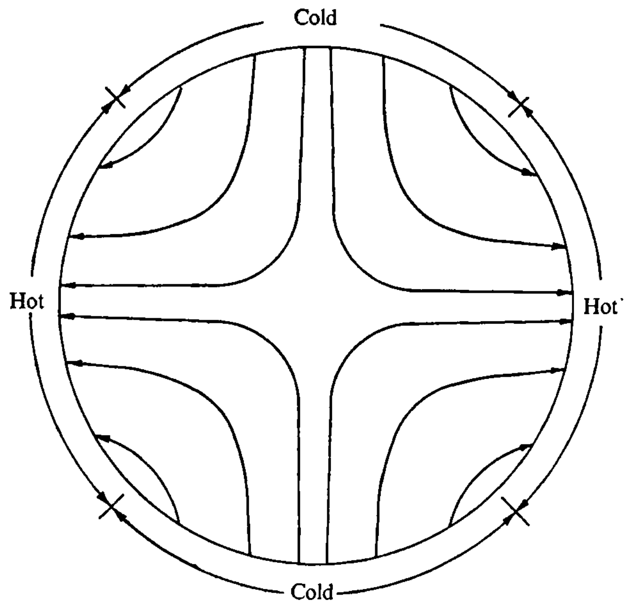


FIGURE 4. Azimuthally-varying boundary temperature producing four-cell flow. Top view of interior streamlines. $T_\Sigma = f(z)\cos 2\theta$.

with boundary condition

$$\frac{\partial \phi_1}{\partial r} = \frac{f'(z)}{\sqrt{2}}\cos(\alpha\theta) \quad \text{on} \quad r = \frac{R}{L}. \tag{90}$$

The solution to this problem can easily be obtained by separation of variables as

$$\phi_1 = \frac{f'(z)}{\sqrt{2}\,\alpha}\frac{r^\alpha}{(R/L)^{\alpha-1}}\cos(\alpha\theta). \tag{91}$$

From this the velocity components can be found by differentiation. The streamlines are given by

$$r = \text{const}/(\sin\alpha\theta)^{1/\alpha}. \tag{92}$$

Then, e.g. when $\alpha = 1$, $f(0) = 0$ and $f'(z)$ is positive, we have the qualitative temperature profile and flow pattern shown in figure 3. We see that in this case a fluid particle rises in the boundary layer near the hot wall, crosses the top of the container in a horizontal boundary layer which arises to conserve mass globally, descends in the boundary layer near the cold wall until it reaches its original level, then crosses the interior of the container from the cold wall to the hot wall in a straight-line motion often referred to as 'plugging'. This preference for purely horizontal motion in the interior (plugging) is often observed in steady natural-convection problems. The cross-flow pattern is shown in figure 4 for $\alpha = 2$ and $f'(z)$ positive. As the parameter $\alpha$ increases through the integers, the flow will continue to divide into $2\alpha$ cells, in order for fluid to enter the interior at a cold wall and leave at a hot one. Notice that, if $f'(z)$ is zero (i.e. the boundary temperature varies only azimuthally and not with $z$), then not only is there no vertical motion, but the cross-flow is also zero, as seen from (91). Although the form of the azimuthally-varying boundary condition used in this example ($T_\Sigma = f(z)\cos\alpha\theta$) appears at first glance to be rather restrictive, it is actually quite general, in that any arbitrary boundary condition can be Fourier synthesized by an infinite series composed of terms of this kind.

It is worth noting that the general solution to the cross-flow problem can be written in terms of a Neumann function, which depends on only the geometry of the boundary curve $\mathscr{C}$. This solution is

$$\phi_1 = \oint_{\mathscr{C}} \frac{\partial \phi_1}{\partial n^*} N\, ds, \tag{93}$$

where $N$ is the Neumann function. The singular Neumann function must be determined from the problem

$$\nabla^2 N = -\frac{\partial W_1}{\partial z} N, \quad \frac{\partial N}{\partial n^*} = 0 \quad \text{on} \quad \mathscr{C}, \qquad (94), (95)$$

$$N \sim -\frac{1}{2\pi} \log |\mathbf{r} - \boldsymbol{\zeta}| \quad \text{as} \quad \mathbf{r} \to \boldsymbol{\zeta}. \tag{96}$$

Here $\mathbf{r}$ is the position vector as usual, and $\boldsymbol{\zeta}$ is the position vector at a field point.

Thus, in principle, the solution for any geometry and any temperature perturbation is given by (93) since $\partial\phi_1/\partial n^*$ on $\mathscr{C}$ is known from (72) and the Neumann function can be found for a particular container geometry from the above problem. Of course, the Neumann function can be found analytically in very few cases and our discussion of simplified geometries and boundary conditions will continue. However, this general solution to the cross-flow problem may be useful in determining numerical solutions to more complicated problems and certainly emphasizes the elliptic behaviour of the cross-flow.

### 4.3. *Right elliptical vertical cylinders*

Let the cross-section of the right vertical cylinder of height $L$ now be an ellipse with semi-major axis $a$ and semi-minor axis $b$. The non-dimensional cross-plane area $A$ and its circumference $\mathscr{C}$ are

$$A = \pi \frac{a}{L}\frac{b}{L}, \tag{97}$$

$$\mathscr{C} = 4\frac{a}{L}\int_0^{\frac{1}{2}\pi} \{1 - [1-(b/a)^2]\sin^2\theta\}^{\frac{1}{2}}\, d\theta = 4\frac{a}{L} E([1-b^2/a^2]^{\frac{1}{2}}), \qquad (98), (99)$$

where $E$ is the complete elliptic integral of the second kind. The heat-up time is

$$\tau_H = \frac{\pi}{2\sqrt{2}} Pr \frac{1}{E([1-b^2/a^2]^{\frac{1}{2}})}. \tag{100}$$

With this expression, the lowest-order temperature and vertical velocity can be found from (74) and (75). If we limit our attention to $z$-dependent boundary conditions, the problem for the cross-flow reduces to

$$\nabla_2^2 \phi_1 = \frac{Pr}{\tau_H}\frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H), \tag{101}$$

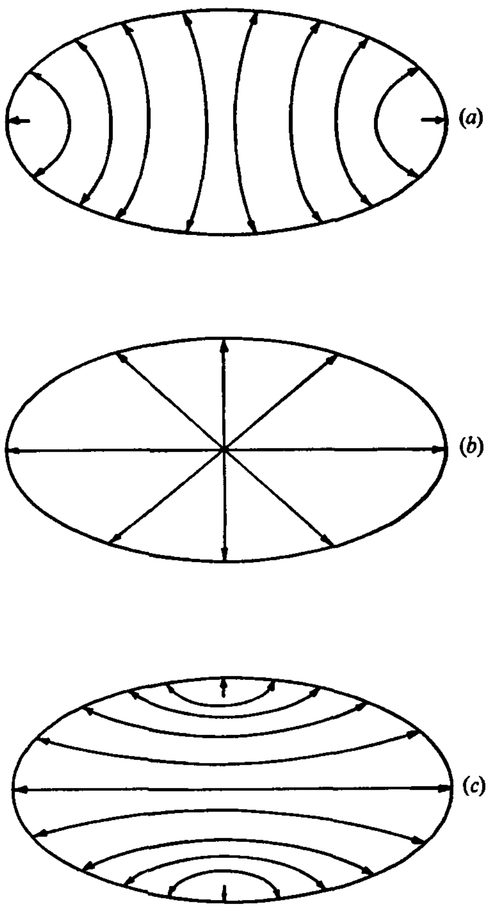


FIGURE 5. Cross-flow streamlines for right elliptical cylinder.
(a) $\mathscr{C} < 2\pi$. (b) $\mathscr{C} = 2\pi$. (c) $\mathscr{C} > 2\pi$.

with boundary condition

$$\frac{\partial \phi_1}{\partial U} = \frac{1}{\sqrt{2}} \frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H) \quad \text{on} \quad U = U_\Sigma. \tag{102}$$

Here $U$ and $V$ are elliptical co-ordinates with $U = U_\Sigma = \tanh^{-1}(b/a)$ on the container. The solution of this equation is

$$\phi_1 = \frac{\mathscr{C}}{\sqrt{2A}} \frac{dT_\Sigma}{dz} \exp(-\tau/\tau_H) \left[\frac{A}{\mathscr{C}}\left(1 - \frac{\mathscr{C}}{2\pi}\right) U + \frac{A}{4\pi} \frac{(\cosh^2 U - \sin^2 V)}{\cosh U_\Sigma \sinh U_\Sigma}\right], \tag{103}$$

where the condition that the cross-plane flow must be everywhere irrotational has been invoked, to render the solution unique. The velocity components follow upon differentiation. The equation for the streamlines may be obtained as

$$\left[\frac{e^{2U} + 2\bar{A} - (4\bar{A}^2 + 1)^{\frac{1}{2}}}{e^{2U} + 2\bar{A} + (4\bar{A}^2 + 1)^{\frac{1}{2}}}\right]^{(4A^{-2}+1)^{-\frac{1}{2}}} = \frac{\text{const.}}{\tan V}, \tag{104}$$

where $$\bar{A} \equiv (2\pi/\mathscr{C} - 1) \cosh U_\Sigma \sinh U_\Sigma. \tag{105}$$

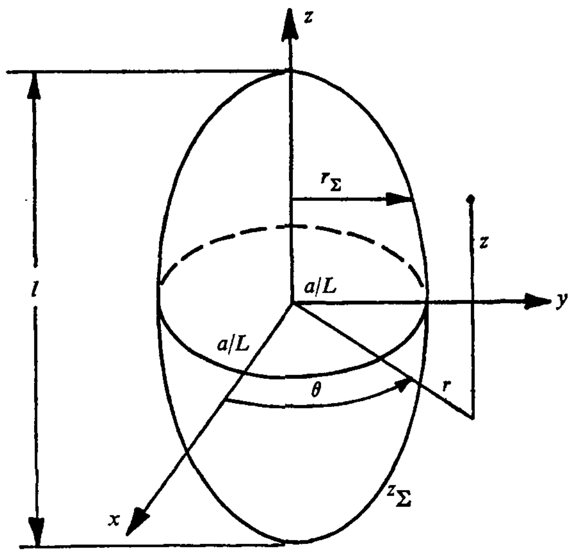


FIGURE 6. Ellipsoid of revolution.

Notice that we have the three possibilities

$$\bar{A} > 0 \quad \text{for} \quad \mathscr{C} < 2\pi, \qquad \bar{A} = 0 \quad \text{for} \quad \mathscr{C} = 2\pi, \qquad \bar{A} < 0 \quad \text{for} \quad \mathscr{C} > 2\pi.$$

Figure 5 shows the cross-flow streamline patterns for each of these three cases. When $\mathscr{C}$ is less than $2\pi$, the streamlines tend to run parallel to the minor axis. When $\mathscr{C}$ equals $2\pi$, the streamlines are straight lines emanating outward from the cylinder's centre-line. When $\mathscr{C}$ is greater than $2\pi$, the streamlines tend to run parallel to the major axis of the ellipse. Inspection of (99) shows that $\mathscr{C}$ equals $2\pi$ when $E([1-b^2/a^2]^{\frac{1}{2}}) = \pi a/2L$. Thus, the three separate cases arise because of a geometry effect, which compares the size of the elliptical cross-section with the height of the cylinder.

### 4.4. *Ellipsoid of revolution*

We now consider the container as an ellipsoid of revolution of height 1 and radius $a$, as shown in figure 6. We shall introduce cylindrical polar co-ordinates, and take the radial co-ordinate $r$ to be $r_\Sigma$ on the container. The non-dimensional cross-plane area and circumference are given by

$$A = \pi r_\Sigma^2, \quad \mathscr{C} = 2\pi r_\Sigma, \quad \text{where} \quad r_\Sigma = a/L(1-4z^2)^{\frac{1}{2}}. \qquad (106)\text{–}(108)$$

The heat-up time, as given by (62), is

$$\tau_H = \frac{Pr\,a}{\sqrt{2}\,L} \frac{(1-4z^2)^{\frac{3}{4}}}{[1-4(1-4(a/L)^2)z^2]^{\frac{1}{4}}}. \qquad (109)$$

We see, in contrast to the cylindrical case, that the ellipsoidal heat-up time is not constant, but varies with $z$.

If we consider the special case of the $z$-dependent boundary condition, then the Poisson equation for flow in the cross-plane is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_1}{dr}\right) = \frac{Pr}{\tau_H}\frac{dT_\Sigma}{dz}\exp\left(-\tau/\tau_H\right), \tag{110}$$

with boundary condition

$$\frac{d\phi_1}{dr} = \frac{\tau_\Sigma}{2}\frac{Pr}{\tau_H}\frac{dT_\Sigma}{dz}\exp\left(-\tau/\tau_H\right) \quad \text{on} \quad r = r_\Sigma. \tag{111}$$

The solution of this problem is

$$u_1 = \frac{Pr}{2}\frac{d}{dz}\left(\frac{T_\Sigma \exp\left(-\tau/\tau_H\right)}{\tau_H}\right)r, \quad v_1 = 0. \qquad (112), (113)$$

The heat-up time for a right circular cylinder of height $L$ and radius $R$ is given by (80). Thus, if we use (109) to calculate the heat-up time for an ellipsoid of revolution of height $L$ and radius $R$, we can form the comparison

$$\frac{\tau_{H(\text{ellipsoid})}}{\tau_{H(\text{cylinder})}} = \frac{(1-4z^2)^{\frac{3}{4}}}{[1-4(R/L)^2z^2]^{\frac{1}{4}}}, \tag{114}$$

which is less than unity, except at $z = 0$. Thus, except at $z = 0$, a right circular cylinder of radius $R$ and length $L$ takes longer to heat-up than the ellipsoid of revolution it circumscribes.

### 4.5. *Remarks*

The general heat-up solution will combine all of the features illustrated in the preceding examples. When an arbitrarily-shaped closed container is perturbed by a general impulsive boundary temperature, a boundary layer will be formed almost instantaneously. This boundary layer will entrain fluid from the inviscid interior, establishing boundary-layer suction, and consequently (because of mass conservation) vertical motion in the interior. Furthermore, the boundary-layer averages azimuthal variations in the container temperature, such that it presents an effective isothermal boundary to each horizontal layer of interior fluid. However, each horizontal layer of fluid in the interior is initially isothermal, since the fluid originally is in a stratified state of static equilibrium. There is no mechanism for heat transfer in the interior, other than convection. Thus, each horizontal layer of isothermal fluid will remain isothermal as it is convected to its new equilibrium position at a velocity whose vertical component is constant across each horizontal layer. The final heated state is approached asymptotically in time; and the $e$-folding time for this heat-up process will vary with vertical location within the container. Finally, the interior fluid layer will (asymptotically) reach a level where its temperature equals the average container temperature, and the fluid will have returned to thermal equilibrium with the container. 'Net' boundary-layer entrainment will cease, as will vertical motion in the interior, and the fluid will be 'heated'. Horizontal motion (plugging) will persist, with fluid entering the interior at a 'cold' wall, and leaving the interior at a 'hot' one, in such a way that the net mass flux into the interior is zero.

## 5. Concluding remarks

The linearized solution to the problem of heat-up from rest has been found. That is, given that a Newtonian, weakly-stratified fluid with constant fluid properties is initially at rest in a completely filled closed container of general shape (ignoring containers with sharp corners and discontinuities in cross-sectional area, which give rise to internal shear layers), and given that the temperature of this container is impulsively changed by a 'very small' amount, then the response of the fluid to this temperature perturbation is known. The interior temperature and vertical velocity solutions are simply written in terms of the circumferential 'average' value of the temperature perturbation. The flow in the cross-plane must be determined by solving a Poisson equation for the particular temperature perturbation and container geometry being considered. This problem is well posed, and numerical solution is straightforward.

It was found, from the analysis, that the inviscid interior region responds to a special 'average' value of the temperature perturbation on the container, and that the effect of the boundary layer is to smear out, or average, any circumferential variation in this perturbation, so that the interior region, in effect, responds to an isothermal cross-plane boundary.

The heat-up mechanism is convective in nature. Conduction and viscosity are important only in thin boundary layers of thickness of the order of $Ra^{-\frac{1}{4}}$ that lie near the container walls. These boundary layers become fully developed within a few periods of the Brunt–Väisälä frequency, then change very slowly during heat-up. The viscous boundary layer requires that a small mass flux be established in the interior region normal to the container side walls, which in turn requires a small vertical mass flow in the interior to preserve continuity. This boundary-layer 'suction' provides the basic heat-up mechanism. By this process, each interior fluid particle convects its 'temperature' (more precisely, its static enthalpy) from its original equilibrium location to some new equilibrium location within the container, where this temperature must necessarily equal the corresponding boundary temperature. Thus, the fluid is heated, i.e. the interior temperature equals the 'boundary' temperature (which is a boundary-layer-averaged container temperature) and the vertical motion ceases. Horizontal motion will persist if the container temperature has azimuthal variations. This is the 'plugging' effect that is common in stratified flows.

Several analytical solutions to the Poisson equation for flow in the cross-plane were found, to illustrate the basic heat-up process, and the alterations to this process that various combinations of the temperature perturbation and container geometry cause. This first calculation for the circular cylinder with a $z$-dependent boundary condition displays all of the physical ideas associated with the general heat-up problem, and at the same time affords great mathematical simplification.

The circular cylinder with an azimuthally-varying boundary condition demonstrates explicitly the concept that the fluid responds to the 'average' temperature, by showing that the interior temperature does not change for a sinusoidal azimuthal perturbation, since the average value of this perturbation is zero.

Furthermore, we see that the cross-flow is not zero for this case; instead, it is the familiar 'plugged' flow found in many stratified-fluid problems.

The elliptical cylinder exhibits an odd 'change of preference for flow direction' dictated by a parameter which compares the (normalized) cross-sectional area of the cylinder with its (normalized) perimeter. The calculation for the ellipsoid primarily demonstrates the fact that the heat-up time varies with vertical position for non-cylindrical containers.

The general container was found to approach its final steady state asymptotically in time, thus the 'heat-up' time was defined to be the $e$-folding time, whose value was established to be proportional to $Ra^{\frac{1}{4}}$. Therefore, referring to (62), the insulating air gap in a pane of thermal glass heats up in approximately 2 s, a 2 ft radius LOX fuel tank in a spacecraft on the pad heats up in 2 h, a one hundred foot diameter LNG storage tank heats up in about 2 days, and the mantle of the earth heats up in from $10^8$ to $10^9$ years. The calculation for the earth's mantle is based on data that are sketchy at best. Furthermore, the Prandtl number in the core is very large, and consequently the viscous dissipation term might not be negligible, as was assumed in our theory. Nonetheless, our calculation shows the possibility that the motion in the earth's mantle may not have reached steady state.

In our work we have assumed that the temperature on the container changes impulsively in time from a function that varies with location on the container only (the basic stratification) to some new function that also depends on only position, as seen from (9). In other words, the boundary temperature is assumed to be independent of time, except during that initial instant in which it is impulsively perturbed. Nevertheless, if the boundary temperature varies slowly in time (on the heat-up time scale of order $Ra^{\frac{1}{4}}$), then the solution to this more general problem may be found from our theory, by treating this time-varying boundary condition in terms of a superpositional integral. This more general solution is possible since the governing equations are linear (hence superposition is valid), and since a boundary condition that varies only on the slow heat-up time scale will drive a flow that is adequately described in terms of this slow time variation. Of course, boundary conditions that vary on a shorter time scale (of the order of the Brunt–Väisälä frequency) cannot be treated by this method, and further investigation is necessary in this case.

The authors gratefully acknowledge the helpful comments of the referees, especially with regard to the arguments concerning the limitations on the parameter $\epsilon$.

## REFERENCES

BARCILON, V. & PEDLOSKY, J. 1967 Linear theory of rotating stratified fluid motions. *J. Fluid Mech.* **29**, 1.

CRABTREE, L. F., KÜCHEMANN, D. & SOWERBY, L. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead) ch. 8) Oxford University Press.

DOTY, R. T. 1973 Linearized buoyant motion in a closed container. Ph.D. thesis, University of Oklahoma.

DOTY, R. T. & JISCHKE, M. C. 1974 Linearized buoyant motion due to impulsively heated vertical plate(s). *Int. J. Heat Mass Transfer*, **16**, 1716.

GREENSPAN, H. P. 1965 On the general theory of contained rotating fluid motions. *J. Fluid Mech.* **22**, 449.

GREENSPAN, H. P. 1969 *The Theory of Rotating Fluids*. Cambridge University Press.

GREENSPAN, H. P. & HOWARD, L. N. 1963 On a time-dependent motion of rotating fluid. *J. Fluid Mech.* **17**, 385.

OSTRACH, S. 1972 *Advances in Heat Transfer* (ed. J. P. Hartnett and T. F. Irvine), vol. 8. Academic.

SAKURAI, T. & MATSUDA, T. 1972 A temperature adjustment process in a Boussinesq fluid via a buoyancy-induced meridional circulation. *J. Fluid Mech.* **54**, 419.

SIEGMANN, W. L. 1971 The spin-down of rotating stratified fluids. *J. Fluid Mech.* **47**, 689.

VERONIS, G. 1970 The analogy between rotating and stratified fluids. *Ann. Rev. Fluid Mech.* **2**, 37.